Harmonic and Anharmonic Oscillator

1. Introduction

A vector field on the circle is described by the equation:

$\dot{\theta} = f(\theta)$ *where: θ – is a point on the circle, $\dot{\theta}$ – is the velocity vector at that point*

A vector field on the circle has different kind of differential equations and its corresponding phase space than vector field on the line to consider. The difference between vector field on the circle and on the line is that in this first case a particle can eventually return to its starting place. Vector fields on the circle provide the most basic model of systems that can oscillate.

Geometric definition: a vector field on the circle is a rule that assigns a unique velocity vector to each point on the circle.

2. Harmonic Oscillator

The simplest oscillator is that one where the phase θ changes uniformly:

$\dot{\theta} = \omega$ $\quad$ $where\ \omega = const$

The solution:

$\theta(t) = \omega t + \theta_0$

The solution corresponds to uniform motion around the circle at an angular frequency ω. The function $\theta(t)$ changes by 2π which means that returns to the same point on the circle, after a time $T = 2\pi/\omega$ – due to that, the solution is periodic. The time T is the period of the oscillation. The period of the oscillation in harmonic oscillator does not depend on the amplitude.

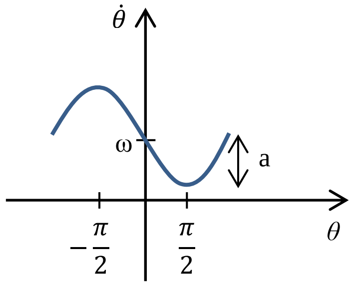
3. Anharmonic Oscillator

To consider the equation:

$\dot{\theta} = \omega - a \sin\theta$

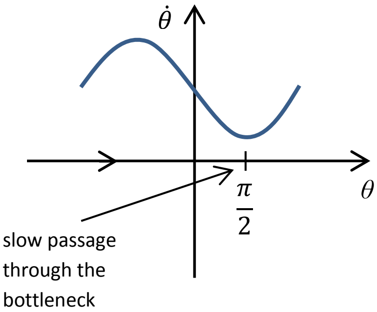
it is convenient to assume that $\omega > 0$ and $a \geq 0$, because for $\omega < 0$ and $a \leq 0$, the situation is similar.

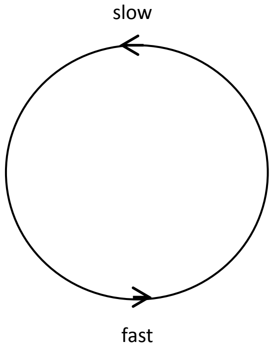
A typical graph of the equation above:



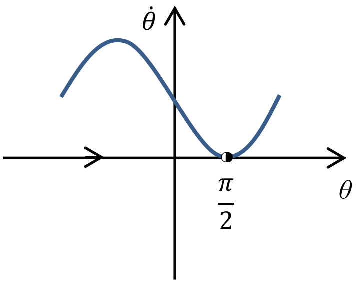
If a = 0, the equation becomes reduced to the equation of harmonic oscillator. The typical graph shows a behavior of "nonuniformity" in the flow around the circle, which is represented by parameter **a.** If a increases, then the nonuniformity becomes more pronounced. There are three cases, that can be considered about connection the parameter a and an angular frequency ω.

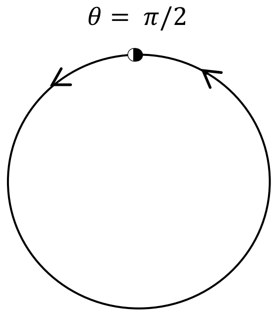
1. $a < \omega \rightarrow$ the oscillation becomes rugged: it takes a long time for the phase point $\theta(t)$ to pass through a bottleneck near the point where $\theta = \pi/2$, and after that it zips around the rest of the circle on a much faster time scale. Below this information is shown on graph and by plotting the vector fields on the circle:



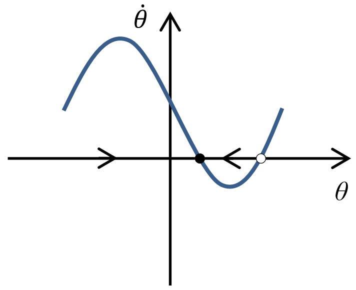


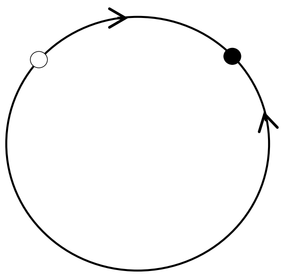
2. $a = \omega \rightarrow$ the oscillation stop entirely: a half-stable fixed point has been born in a saddle-node bifurcation at $\theta = \pi/2$.





3. $a < \omega \rightarrow$ the half-stable fixed point divides into a stable and unstable fixed point.





(All trajectories are attracted to the stable fixed point as $t \rightarrow \infty$).

Oscillation period

For the first case ($a < \omega$) the period of oscillation can be found analytically:

$$T = \int dt = \int_0^{2\pi} \frac{dt}{d\theta} d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a \sin\theta}$$

To replace *dt/dθ* used the equation $\dot{\theta} = \omega - a \sin\theta$. This integral can be resolved by the substitution (u = tan*θ/2*) or by complex variable methods, giving the result:

$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

For the second case (a = *0*) the equation of oscillation period reduces to:

$$T = \frac{2\pi}{\omega}$$

which is familiar to the oscillation period for harmonic oscillator.
The period increases with a and diverges as a approaches ω from below (this limit is called as $a \to \omega^-$). The order of the divergence can be estimated by noting that:

$$\sqrt{\omega^2 - a^2} = \sqrt{\omega + a} \cdot \sqrt{\omega - a} \approx \sqrt{2\omega} \cdot \sqrt{\omega - a}$$

Then:

$$T \approx \left(\frac{\pi\sqrt{2}}{\sqrt{\omega}}\right) \cdot \frac{1}{\sqrt{\omega - a}}$$

which shows that T blows up like $(a_c - a)^{-\frac{1}{2}}$, where $a_c = \omega$.